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Pseudo-gradient and Lagrangian boundary control system formulation of electromagnetic fields

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Abstract
This paper describes an electromagnetic field analogue of the classical Brayton–Moser formulation. It is shown that Maxwell's curl equations constitute a pseudo-gradient system with respect to a single electromagnetic mixed-potential functional and a metric defined by the constitutive relations of the fields. Besides its use for the generation of power-based Lyapunov functionals for stability analysis and Poynting-like flow balances, the electromagnetic mixed-potential formulation suggests a family of alternative variational principles. This yields a novel Lagrangian boundary control system formulation admitting nonzero energy flow through the boundary. The corresponding symplectic Hamiltonian system is still associated with the total electromagnetic field energy.

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1. Introduction and motivation

In the early 1960s, Moser developed a mathematical analysis to study the stability of electrical networks containing tunnel diodes [19]. His method is based on the introduction of a scalar function of which the gradient leads to the equations of motion for the nonlinear network. Four years later, Moser generalized the method together with Brayton in the centennial paper [5] and showed that the dynamics of a broad class of (possibly nonlinear) RLC networks can be written in the form

$$\mathbf{M}(\boldsymbol{u})\dot{\boldsymbol{u}} = \mathcal{P}_{\boldsymbol{u}}(\boldsymbol{u}), \tag{1}$$

where $\boldsymbol{u} = \mathrm{col}(\boldsymbol{i}, \boldsymbol{v}) \in \mathbb{R}^n$ is a vector containing the inductor currents $\boldsymbol{i} \in \mathbb{R}^{n_L}$ and the capacitor voltages $\boldsymbol{v} \in \mathbb{R}^{n_C}$, respectively, with $n = n_L + n_C$ the number of inductors and capacitors in the network, and the matrix $\mathbf{M} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ defining an indefinite (pseudo-Riemannian) metric containing the incremental inductances and capacitances of the form

$$\mathbf{M}(\boldsymbol{i}, \boldsymbol{v}) = \begin{pmatrix} -\mathbf{L}(\boldsymbol{i}) & \mathbf{0} \\ \mathbf{0} & \mathbf{C}(\boldsymbol{v}) \end{pmatrix}. \tag{2}$$

The scalar function $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}$ represents the so-called mixed-potential function describing the resistive and the topological structure of the network. The subscript notation $(\cdot)_{\boldsymbol{u}}$ denotes partial differentiation with respect to $\boldsymbol{u}$. In case of a topologically complete[3] network the mixed-potential function takes the form

$$\mathcal{P}(\boldsymbol{i}, \boldsymbol{v}) = \mathcal{N}(\boldsymbol{i}, \boldsymbol{v}) + \mathcal{R}(\boldsymbol{i}) - \mathcal{G}(\boldsymbol{v}), \tag{3}$$

where $\mathcal{N}(\boldsymbol{i}, \boldsymbol{v}) = \boldsymbol{i} \cdot \boldsymbol{\gamma v}$, with constant matrix $\boldsymbol{\gamma} \in \mathbb{R}^{n_L \times n_C}$, is determined by the interconnection of the inductors and capacitors. The functions $\mathcal{R}(\boldsymbol{i}) = \int \boldsymbol{v}_r(\boldsymbol{i}) \cdot \mathrm{d}\boldsymbol{i}$ and $\mathcal{G}(\boldsymbol{v}) = \int \boldsymbol{i}_g(\boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{v}$ denote the current and voltage potentials (or content and co-content [15]) which are related to the current-controlled resistors and voltage sources, and the voltage-controlled resistors and current sources, respectively. Equations of the form (1), together with (2) and (3), are known as the Brayton–Moser (BM) equations and constitute Kirchhoff's loop and node equations for the network. Observe that (3) has units akin to power.

1.1. Application of the BM framework

The principal application of the concept of mixed potential-concerns its use in determining (Lyapunov-based) stability criteria for nonlinear networks. This is motivated by the fact that the time derivative of (3) along the trajectories of (1) is a quadratic form in $\dot{\boldsymbol{u}}$, i.e.,

$$\dot{\mathcal{P}}(\boldsymbol{u}) = \dot{\boldsymbol{u}} \cdot \mathbf{M}(\boldsymbol{u})\dot{\boldsymbol{u}}. \tag{4}$$

If the network only contains resistors, inductors and sources (RL network), then $\boldsymbol{u} = \boldsymbol{i}$ and $\mathbf{M}(\boldsymbol{u}) = -\mathbf{L}(\boldsymbol{i})$, with $\mathbf{L}(\boldsymbol{i}) > 0$ as is the case usually. Hence, the mixed-potential $\mathcal{P}(\boldsymbol{i}) = \mathcal{R}(\boldsymbol{i})$ is non-increasing since $\dot{\mathcal{P}}(\boldsymbol{i}) \leqslant 0$, which implies that the network is stable.

However, in general, the right-hand side of (4) is indefinite. To circumvent this problem, Brayton and Moser proposed a family of alternative pairs, say $\widetilde{\mathcal{P}}$ and $\widetilde{\mathbf{M}}$, that preserve the network dynamics (1), i.e., $\widetilde{\mathbf{M}}^{-1}(\boldsymbol{u})\widetilde{\mathcal{P}}_{\boldsymbol{u}}(\boldsymbol{u}) = \mathbf{M}^{-1}(\boldsymbol{u})\mathcal{P}_{\boldsymbol{u}}(\boldsymbol{u})$, and such that

$$\dot{\widetilde{\mathcal{P}}}(\boldsymbol{u}) = \dot{\boldsymbol{u}} \cdot \widetilde{\mathbf{M}}(\boldsymbol{u})\dot{\boldsymbol{u}} \leqslant 0. \tag{5}$$

This observation has led to the construction of several stability theorems, each depending on different specific properties of the network [5, 14].

1.2. Contribution and outline

During the last four decades several notable extensions and generalizations of the Brayton–Moser formulation have been presented in the literature. However, to our knowledge, all these contributions consider lumped-parameter networks only (see, e.g., [12] and the references therein) except for the work contained in [4]. The latter paper presents a mixed-potential-based stability theory of a single transmission line system connected to a nonlinear load. Instead of a mixed-potential function (3), the analysis involves the construction of a mixed-potential functional which is defined along the spatial domain and at both the end points (boundaries) of the transmission line.

[3] A network is said to be topologically complete if Kirchhoff's laws are expressible solely in terms of the inductor currents and the capacitor voltages.

In this paper, we introduce an electromagnetic analogue of the results presented in [4, 5]. This allows us to formulate Maxwell's curl equations (i.e., the Ampère–Maxwell and Faraday's law) in a manner analogous to (1). Apart from providing an elegant canonical form or the generation of (power-like) Lyapunov functions for stability analysis, this opens up a way for the application of power-based passivity and control techniques [13, 23] in the realm of electromagnetic field systems, or distributed-parameter systems in general.

Furthermore, the construction of an electromagnetic mixed-potential also gives rise to some alternative variational principles that imply the existence of a family of novel Lagrangian functionals. In contrast to the existing methods in mathematical physics, these Lagrangian functionals explicitly yield *both* Maxwell's curl equations explicitly without the usual restriction to systems with infinite spatial domain, where the dynamic variables go to zero for the spatial variables tending to infinity, and without imposing the usual assumption that the energy exchange through the boundary is zero [3, 22]. This leads to the definition of an infinite-dimensional Lagrangian boundary control system, as originally introduced for mechanical systems in [6] (see also [21] for a summary and further developments on the topic). Although the new Lagrangians do not possess the usual electric minus magnetic energy structure (like kinetic minus potential energy in the mechanical case), the associated (symplectic) Hamiltonian counter part is shown to coincide with the total energy stored by the fields. Additionally, the framework avoids the introduction of the usual vector magnetic and scalar electric potentials. Hence, the construction of a Lagrangian or Hamiltonian functional in situations where it is desirable to incorporate the magnetic charge densities [9, 16] implying $\operatorname{div} \mathbf{B} \neq 0$ is straightforward.

The remainder of the paper is organized as follows. Section 2 briefly reviews the Maxwell equations and some of its properties. The main result is presented in section 3. First, in subsection 3.1, an electromagnetic mixed-potential formulation is constructed for lossless systems with zero boundary energy flow. The formulation is generalized to lossy electromagnetic systems with nonzero boundary energy flow in subsections 3.2 and 3.3. Section 4 presents a brief outline on how the electromagnetic mixed-potential can be used for stability analysis and the construction of Poynting-like flow balances. A set of alternative Lagrangian and associated Hamiltonian functionals is proposed in section 5. In subsection 5.1, the framework is again first constructed for lossless systems with zero boundary energy flow. The Hamiltonian counterpart is treated in subsection 5.2. The result is generalized to lossy electromagnetic systems with nonzero boundary energy flow in subsection 5.3. Finally, in subsection 5.4, the new formulations are compared to the classical approach. The paper is concluded with some final remarks and topics for future research.

2. Maxwell's equations

The equations governing the electromagnetic behaviour in a medium $\mathbb{V}$ can be split into two subsets [11]: Maxwell's 'curl' equations

$$\operatorname{curl} \mathbf{E} = -\mathbf{B}_t \tag{6}$$

$$\operatorname{curl} \mathbf{H} = \mathbf{D}_t + \mathbf{J} \tag{7}$$

and Maxwell's 'divergence' equations

$$\operatorname{div} \mathbf{D} = \rho \tag{8}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{9}$$

in which the flux densities $\mathbf{B}$ and $\mathbf{D}$ are often related to the field intensities $\mathbf{H}$ and $\mathbf{E}$ through the constitutive relationships depending on the medium. The vector $\mathbf{J}$ denotes the current density and ρ denotes the electric charge density. Furthermore, if $\mathbf{H}^{\text{ext}}$ and $\mathbf{E}^{\text{ext}}$ represent some external fields and if $\mathbf{J}^{\text{s}}$ represents a current sheet along the surface of the medium, then the associated boundary conditions for the tangential magnetic fields are given by

$$\hat{\mathbf{n}} \times (\mathbf{H} - \mathbf{H}^{\text{ext}}) = \mathbf{J}^{\text{s}}, \tag{10}$$

where $\hat{\mathbf{n}}$ is the inward normal, while for the tangential we have the electric fields

$$\hat{\mathbf{n}} \times (\mathbf{E} - \mathbf{E}^{\text{ext}}) = \mathbf{0}. \tag{11}$$

The boundary conditions of the normal components of the fields read $\hat{\mathbf{n}} \cdot (\mathbf{D} - \mathbf{D}^{\text{ext}}) = \rho^{\text{s}}$, where ρ^{s} represents a surface charge and $\hat{\mathbf{n}} \cdot (\mathbf{B} - \mathbf{B}^{\text{ext}}) = \mathbf{0}$.

A very important property in the study of electromagnetic fields is Poynting's theorem:

$$\int_{\mathbb{V}} (\mathbf{H} \cdot \mathbf{B}_t + \mathbf{E} \cdot \mathbf{D}_t)\, \mathrm{d}v = \int_{\partial\mathbb{V}} \mathbf{S} \cdot \hat{\mathbf{n}}\, \mathrm{d}s - \int_{\mathbb{V}} \mathbf{J} \cdot \mathbf{E}\, \mathrm{d}v, \tag{12}$$

with $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ denoting the Poynting vector. Often it is convenient to associate the left-hand side with the time derivative of the electromagnetic energy density $\bar{\mathcal{E}}_t = \mathbf{H} \cdot \mathbf{B}_t + \mathbf{E} \cdot \mathbf{D}_t$. However, the explicit expression for $\bar{\mathcal{E}}$ depends on the constitutive relationships between $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$, $\mathbf{E}$ (see, e.g., [10, 20]).

3. Electromagnetic mixed-potential

This section shows how to cast Maxwell's curl equations (6) and (7) into a form similar to (1). The network concepts as presented in section 1 will be used as the guiding thoughts of the discussion. The first analogy we pose is the selection of the field intensities $\mathbf{H}$ and $\mathbf{E}$ to play a role similar to the currents and voltages in (1). For the ease of presentation we will first assume that the medium is linear, time- and space-invariant, such that $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \varepsilon\mathbf{E}$, where the scalars μ and ε represent the permeability and the permitivity of the medium, respectively. (This assumption will be relaxed in remark 2.) Hence, we will focus on developing a Brayton–Moser (BM) analogue of

$$\mu\mathbf{H}_t = -\mathrm{curl}\,\mathbf{E} \tag{13}$$

$$\varepsilon\mathbf{E}_t = \mathrm{curl}\,\mathbf{H} - \mathbf{J}. \tag{14}$$

The next main step in completing the analogy concerns the construction of a functional in which we can identify similar terms as in (3), that is, we want to identify with (13) and (14) a functional of the form

$$\mathcal{P}(\mathbf{E}, \mathbf{H}) = \mathcal{N}(\mathbf{H}, \mathbf{E}) + \mathcal{R}(\mathbf{H}) - \mathcal{G}(\mathbf{E}). \tag{15}$$

The derivatives of the latter with respect to the field intensities are determined by invoking the functional (or variational) derivative playing a role analogous to the gradient of a function. Let us first consider the case in which the medium is loss- and sourceless, i.e., $\mathbf{J} = \mathbf{0}$, and the energy through the boundary is zero.

3.1. Zero boundary energy flow

Let $\mathcal{Q}(\mathbf{v})$ be a functional of a smooth vector field $\mathbf{v}(\mathbf{r}, t) \in \mathbb{R}^3$, with $\mathbf{r} = (x, y, z)$. The functional derivative, denoted by $\delta_{\mathbf{v}}\mathcal{Q}(\mathbf{v})$, is uniquely defined from [7]

$$\int_{\mathbb{V}} \delta_{\mathbf{v}}\mathcal{Q}(\mathbf{v}) \cdot \mathbf{w}\, \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}\eta}\mathcal{Q}(\mathbf{v} + \eta\mathbf{w})|_{\eta=0}, \tag{16}$$

for any $\eta \in \mathbb{R}$ and any smooth test field $\mathbf{w}(\mathbf{r}, t) \in \mathbb{R}^3$ such that $\mathbf{v} + \eta\mathbf{w}$ satisfies the same boundary conditions as $\mathbf{v}$. As will be shown, under certain boundary conditions, for Maxwell's curl equations (13) and (14) it suffices to consider functionals of the form

$$\mathcal{Q}(\mathbf{v}) = \int_{\mathbb{V}} \bar{\mathcal{Q}}(\mathbf{v}, \operatorname{curl} \mathbf{v})\, \mathrm{d}v, \tag{17}$$

with density $\bar{\mathcal{Q}}(\mathbf{v}, \operatorname{curl} \mathbf{v})$. By Taylor's theorem

$$\begin{aligned}
\mathcal{Q}(\mathbf{v} + \eta\mathbf{w}) &= \mathcal{Q}(\mathbf{v}) + \eta\mathcal{Q}'(\mathbf{v}, \mathbf{w}) + \mathcal{O}(\eta^2) \\
&= \mathcal{Q}(\mathbf{v}) + \eta \int_{\mathbb{V}} (\bar{\mathcal{Q}}_{\mathbf{v}} \cdot \mathbf{w} + \bar{\mathcal{Q}}_{\operatorname{curl}\mathbf{v}} \cdot \operatorname{curl} \mathbf{w})\, \mathrm{d}v + \mathcal{O}(\eta^2) \\
&= \mathcal{Q}(\mathbf{v}) + \eta \int_{\mathbb{V}} (\bar{\mathcal{Q}}_{\mathbf{v}} + \operatorname{curl} \bar{\mathcal{Q}}_{\operatorname{curl}\mathbf{v}}) \cdot \mathbf{w}\, \mathrm{d}v \\
&\quad - \eta \int_{\mathbb{V}} \operatorname{div}(\bar{\mathcal{Q}}_{\operatorname{curl}\mathbf{v}} \times \mathbf{w})\, \mathrm{d}v + \mathcal{O}(\eta^2),
\end{aligned}$$

where we have used the vector identity $\operatorname{div}(\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot \operatorname{curl} \mathbf{v} - \mathbf{v} \cdot \operatorname{curl} \mathbf{w}$. Furthermore, using Gauss' divergence theorem, we can write

$$\int_{\mathbb{V}} \operatorname{div}(\bar{\mathcal{Q}}_{\operatorname{curl}\mathbf{v}} \times \mathbf{w})\, \mathrm{d}v = \int_{\partial\mathbb{V}} (\hat{\mathbf{n}} \times \bar{\mathcal{Q}}_{\operatorname{curl}\mathbf{v}}) \cdot \mathbf{w}\, \mathrm{d}s,$$

which, under the assumption that at the boundary surface $\hat{\mathbf{n}} \times \bar{\mathcal{Q}}_{\operatorname{curl}\mathbf{v}} = \mathbf{0}$, finally yields that

$$\delta_{\mathbf{v}}\mathcal{Q}(\mathbf{v}) = \bar{\mathcal{Q}}_{\mathbf{v}} + \operatorname{curl}(\bar{\mathcal{Q}}_{\operatorname{curl}\mathbf{v}}). \tag{18}$$

For the case that $\mathcal{Q}(\mathbf{v})$ has an extremal, i.e., when $\delta_{\mathbf{v}}\mathcal{Q}(\mathbf{v}) = \mathbf{0}$, the latter equation coincides with the well-known Euler differential equation [7].

Returning to our goal of finding the electromagnetic analogue of (1), we are now in a position to postulate the following. The assumption that the medium is loss- and sourceless implies that both $\mathcal{R}(\mathbf{H})$ and $\mathcal{G}(\mathbf{E})$ are zero. This means it suffices to consider only a mixed-potential functional of the form $\mathcal{P}(\mathbf{H}, \mathbf{E}) = \mathcal{N}(\mathbf{H}, \mathbf{E})$, with

$$\mathcal{N}(\mathbf{H}, \mathbf{E}) = \int_{\mathbb{V}} \bar{\mathcal{N}}(\bullet)\, \mathrm{d}v. \tag{19}$$

Depending on the boundary conditions, we are left with two possible choices for $\bar{\mathcal{N}}(\bullet)$, which we denote as follows:

$$\bar{\mathcal{N}}(\bullet) = \begin{cases} \bar{\mathcal{N}}^{\mathrm{m}}(\mathbf{H}, \operatorname{curl} \mathbf{E}) = \mathbf{H} \cdot \operatorname{curl} \mathbf{E} \\ \bar{\mathcal{N}}^{\mathrm{e}}(\operatorname{curl} \mathbf{H}, \mathbf{E}) = \operatorname{curl} \mathbf{H} \cdot \mathbf{E}. \end{cases} \tag{20}$$

Indeed, the selection

$$\mathcal{P}(\mathbf{H}, \mathbf{E}) = \int_{\mathbb{V}} \bar{\mathcal{N}}^{\mathrm{m}}(\mathbf{H}, \operatorname{curl} \mathbf{E})\, \mathrm{d}v \tag{21}$$

imposes the condition that the magnetic field intensity at the boundary is continuous, i.e., $\hat{\mathbf{n}} \times \mathbf{H} = \mathbf{0}$, and in turn ensures the set of well-defined functional derivatives

$$\delta_{\mathbf{H}}\mathcal{P}(\mathbf{H}, \mathbf{E}) = \operatorname{curl} \mathbf{E}, \qquad \delta_{\mathbf{E}}\mathcal{P}(\mathbf{H}, \mathbf{E}) = \operatorname{curl} \mathbf{H},$$

as desired. Hence, letting $\mathbf{u} = \operatorname{col}(\mathbf{H}, \mathbf{E})$ represent the field intensity vector, Maxwell's curl equations (13) and (14) define a pseudo-gradient system of the form (cf (1))

$$\mathbf{M}\mathbf{u}_t = \delta_{\mathbf{u}}\mathcal{P}(\mathbf{u}), \tag{22}$$

with respect to the indefinite (pseudo-Riemannian) metric

$$\mathbf{M} = \begin{pmatrix} -\mu\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \varepsilon\mathbf{I}_3 \end{pmatrix}, \tag{23}$$

where $\mathbf{I}_3$ denotes the 3×3 identity matrix.

On the other hand, precisely the same result can be obtained by selecting

$$\mathcal{P}(\mathbf{H},\mathbf{E}) = \int_{\mathbb{V}} \bar{\mathcal{N}}^{\mathrm{e}}(\operatorname{curl}\mathbf{H},\mathbf{E})\,\mathrm{d}v \tag{24}$$

accompanied by the assumption that now the tangential electric field intensity at the boundary is continuous, i.e., $\hat{\mathbf{n}} \times \mathbf{E} = \mathbf{0}$. If the boundary is a perfect conductor the latter condition seems natural but the former condition that $\hat{\mathbf{n}} \times \mathbf{H} = \mathbf{0}$ implies a rather unphysical boundary condition. This can be explained as follows. Assuming $\hat{\mathbf{n}} \times \mathbf{E} = \mathbf{0}$ naturally implies by (11) that the external electric field $\mathbf{E}^{\mathrm{ext}} = \mathbf{0}$. The condition that the tangential component of $\mathbf{H}$ vanishes at the boundary surface also implies that there is no external field influence at the boundary surface of the medium $\mathbf{H}^{\mathrm{ext}} = \mathbf{0}$. However, the tangential component of $\mathbf{H}$ should, in general, be discontinuous by an amount whose magnitude is equal to the magnitude of a surface current sheet $\mathbf{J}^{\mathrm{s}}$ (cf (10)) and whose direction is parallel to $\hat{\mathbf{n}} \times \mathbf{J}^{\mathrm{s}}$ [11]. Another possibility could be that the surface current is precisely 'compensated' by an external field according to $\mathbf{J}^{\mathrm{s}} = -(\hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{ext}})$. Furthermore, it is clear from Poynting's theorem (12) that the energy flow across the boundary depends on both the tangential electric field and the tangential magnetic field. Specification of either $\hat{\mathbf{n}} \times \mathbf{E} = \mathbf{0}$, $\hat{\mathbf{n}} \times \mathbf{H} = \mathbf{0}$, or $\hat{\mathbf{n}} \times \mathbf{H} = \mathbf{J}^{\mathrm{s}}$ implies that the net energy flow across the boundary is zero, i.e.,

$$\int_{\partial\mathbb{V}} \mathbf{S} \cdot \hat{\mathbf{n}}\,\mathrm{d}s = 0, \tag{25}$$

meaning that the system is isolated.

The continuity restrictions, $\hat{\mathbf{n}} \times \mathbf{H} = \mathbf{0}$ or $\hat{\mathbf{n}} \times \mathbf{E} = \mathbf{0}$, are necessary to guarantee well-defined functional derivatives. As suggested in [7], such situations can be remedied by adding appropriate boundary terms to the respective functionals. For the mixed-potential functional this will be illustrated in subsection 3.3. Next, we first consider the case that $\mathbf{J} \neq \mathbf{0}$ which implies the existence of an electromagnetic co-content functional.

3.2. Electromagnetic co-content

If the media are lossy, or contain sources, we may associate with the Ampère–Maxwell law (14) a functional

$$\mathcal{G}(\mathbf{E}) = \int_{\mathbb{V}} \bar{\mathcal{G}}(\mathbf{E},\mathbf{r})\,\mathrm{d}v, \tag{26}$$

with the possibly nonlinear and nonhomogeneous density function

$$\bar{\mathcal{G}}(\mathbf{E},\mathbf{r}) = \int \mathbf{J}(\mathbf{E},\mathbf{r}) \cdot \mathrm{d}\mathbf{E}. \tag{27}$$

The existence of (27) is guaranteed as long as $\mathbf{J}_{\mathbf{E}}(\mathbf{E},\mathbf{r}) = \mathbf{J}_{\mathbf{E}}^{T}(\mathbf{E},\mathbf{r})$, i.e., the resistive structure should be reciprocal. Hence, the mixed-potential functional can be extended to (cf (3))

$$\mathcal{P}(\mathbf{H},\mathbf{E}) = \mathcal{N}(\mathbf{H},\mathbf{E}) - \mathcal{G}(\mathbf{E}), \tag{28}$$

where $\mathcal{N}(\mathbf{H},\mathbf{E})$ is selected from (20). The functional $\mathcal{G}(\mathbf{E})$ clearly plays a role similar to the co-content function in (3). For this reason, we refer to (26) as the total electromagnetic co-content. Observe that if $\mathbf{J}$ represents an independent field of sources the co-content density reduces to $\bar{\mathcal{G}}(\mathbf{E}) = \mathbf{J} \cdot \mathbf{E}$. Linear Ohmic losses are included by selecting $\bar{\mathcal{G}}(\mathbf{E}) = \frac{1}{2}\sigma\|\mathbf{E}\|^2$, where σ represents the specific conductivity.

Remark 1. In the electromagnetic field setting considered in this paper there does not exist an electromagnetic content functional (analogously to $\mathcal{R}(\boldsymbol{i})$ in (3)). However, sometimes it can be useful to consider, next to the electric current density $\mathbf{J}$, a magnetic current density $\mathbf{J}^{\mathrm{m}}$ together with its associated magnetic charge density ρ^{m} such that $\operatorname{div}\mathbf{B} = \rho^{\mathrm{m}}$ [9]. In such a case the electromagnetic content reads

$$\mathcal{R}(\mathbf{H}) = \int_{\mathbb{V}} \mathbf{J}^{\mathrm{m}} \cdot \mathbf{H}\,\mathrm{d}v. \tag{29}$$

The introduction of these quantities puts the field equations in a symmetrical form due to the duality between the electric and magnetic properties of the fields.

3.3. Nonzero boundary energy flow

In order to accommodate our framework for nonzero boundary energy flow and to avoid nonphysical boundary conditions we need to extend the definition of the functional derivative (18). Borrowing inspiration from [4, 7], we define an extended vector field

$$[\mathbf{v}] = \begin{pmatrix} \mathbf{v} \\ \mathbf{v}^{\mathrm{b}} \end{pmatrix},$$

and consider instead of (17) a functional

$$\mathcal{Q}[\mathbf{v}] = \int_{\mathbb{V}} \bar{\mathcal{Q}}(\mathbf{v}, \operatorname{curl}\mathbf{v})\,\mathrm{d}v + \mathcal{Q}^{\mathrm{b}}(\mathbf{v}^{\mathrm{b}}).$$

Here $\mathbf{v}^{\mathrm{b}}$ represents a boundary vector, and $\mathcal{Q}^{\mathrm{b}}(\mathbf{v}^{\mathrm{b}})$ represents a boundary functional

$$\mathcal{Q}^{\mathrm{b}}(\mathbf{v}^{\mathrm{b}}) = \int_{\partial\mathbb{V}} \bar{\mathcal{Q}}^{\mathrm{b}}(\mathbf{v})\,\mathrm{d}s.$$

This means that the functional derivative (18) can be accommodated to deal with nonzero tangential boundary field terms as follows:

$$\delta_{[\mathbf{v}]}\mathcal{Q}[\mathbf{v}] = \begin{pmatrix} \bar{\mathcal{Q}}_{\mathbf{v}} + \operatorname{curl}(\bar{\mathcal{Q}}_{\operatorname{curl}\mathbf{v}}) \\ \bar{\mathcal{Q}}^{\mathrm{b}}_{\mathbf{v}} - \hat{\mathbf{n}} \times \bar{\mathcal{Q}}_{\operatorname{curl}\mathbf{v}} \end{pmatrix}. \tag{30}$$

The second term represents the system's natural boundary conditions and, together with the first term, vanishes at the extremals of $\mathcal{Q}$.

For our purpose here, this suggests that (22) can be extended to

$$[\mathbf{M}][\mathbf{u}_t] = \delta_{[\mathbf{u}]}\mathcal{P}[\mathbf{u}], \tag{31}$$

where $[\mathbf{M}] = \operatorname{diag}(\mathbf{M}, \mathbf{M}^{\mathrm{b}})$. To make this explicit, we proceed as follows. Suppose that we start from $\bar{\mathcal{N}} = \bar{\mathcal{N}}^{\mathrm{m}}$, then

$$\mathcal{P}[\mathbf{H}, \mathbf{E}] = \int_{\mathbb{V}} \mathbf{H} \cdot \operatorname{curl}\mathbf{E}\,\mathrm{d}v - \int_{\mathbb{V}} \mathbf{J} \cdot \mathbf{E}\,\mathrm{d}v + \mathcal{P}^{\mathrm{b}}(\bullet). \tag{32}$$

Then, recalling $\mathbf{u} = \operatorname{col}(\mathbf{H}, \mathbf{E})$, we obtain

$$\delta_{[\mathbf{u}]}\mathcal{P} = \begin{pmatrix} \operatorname{curl}\mathbf{E} \\ \operatorname{curl}\mathbf{H} - \mathbf{J} \\ \bar{\mathcal{P}}^{\mathrm{b}}_{\mathbf{H}} \\ \bar{\mathcal{P}}^{\mathrm{b}}_{\mathbf{E}} - \hat{\mathbf{n}} \times \mathbf{H} \end{pmatrix}. \tag{33}$$

For the electromagnetic mixed-potential (32) the BM formulation can be readily completed by selecting the boundary potential

$$\mathcal{P}^{\mathrm{b}}(\mathbf{E}^{\mathrm{b}}) = \int_{\partial\mathbb{V}} \mathbf{J}^{\mathrm{s}} \cdot \mathbf{E}\,\mathrm{d}s, \tag{34}$$

yielding the boundary condition $\hat{\mathbf{n}} \times \mathbf{H} = \mathbf{J}^{\mathrm{s}}$. However, with this choice we again have that the system is isolated from the rest of space implying that the net energy flow across the boundary is zero (25). A more general setting is obtained by replacing the integrand in (34) with a term $(\mathbf{J}^{\mathrm{s}} + \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{ext}}) \cdot \mathbf{E}$. Observe that since $\mathcal{P}^{\mathrm{b}}$ depends on the electric field intensity at the boundary surface only it belongs to the category of co-content functionals.

On the other hand, starting from $\bar{\mathcal{N}} = \bar{\mathcal{N}}^{\mathrm{e}}$ gives

$$\mathcal{P}[\mathbf{H}, \mathbf{E}] = \int_{\mathbb{V}} \mathbf{E} \cdot \operatorname{curl} \mathbf{H} \, \mathrm{d}v - \int_{\mathbb{V}} \mathbf{J} \cdot \mathbf{E} \, \mathrm{d}v + \mathcal{P}^{\mathrm{b}}(\bullet), \tag{35}$$

with $\mathcal{P}^{\mathrm{b}}(\bullet)$ to be defined and yields the natural boundary conditions:

$$\mathbf{0} = \bar{\mathcal{P}}^{\mathrm{b}}_{\mathbf{H}} - \hat{\mathbf{n}} \times \mathbf{E} \tag{36}$$

$$\mathbf{0} = \bar{\mathcal{P}}^{\mathrm{b}}_{\mathbf{E}}. \tag{37}$$

Clearly, the latter conditions are satisfied by selecting either a boundary (content) potential $\mathcal{P}^{\mathrm{b}}(\mathbf{H}^{\mathrm{b}}) = 0$, yielding the physically admissible condition $\hat{\mathbf{n}} \times \mathbf{E} = \mathbf{0}$, or

$$\mathcal{P}^{\mathrm{b}}(\mathbf{H}^{\mathrm{b}}) = \int_{\partial \mathbb{V}} (\hat{\mathbf{n}} \times \mathbf{E}^{\mathrm{ext}}) \cdot \mathbf{H} \, \mathrm{d}s, \tag{38}$$

which provides the more general boundary condition (11).

We refer to (31), together with either the mixed-potential functional of the form (32) or (35), as the electromagnetic BM equations.

Remark 2. So far we have considered only linear, time- and space-invariant media. However, in general, the constitutive relationships between the flux densities and the field intensities are given by $\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{B}}(\mathbf{H}, \mathbf{E}, \mathbf{r}, t)$ and $\mathbf{D}(\mathbf{r}, t) = \hat{\mathbf{D}}(\mathbf{H}, \mathbf{E}, \mathbf{r}, t)$. Apart from explicit time dependence, the scope of systems that can be described in the present framework extends naturally by noting that $\mathbf{B}_t = \hat{\mathbf{B}}_{\mathbf{H}}(\mathbf{H}, \mathbf{E}, \mathbf{r})\mathbf{H}_t + \hat{\mathbf{B}}_{\mathbf{E}}(\mathbf{H}, \mathbf{E}, \mathbf{r})\mathbf{E}_t$ and $\mathbf{D}_t = \hat{\mathbf{D}}_{\mathbf{H}}(\mathbf{H}, \mathbf{E}, \mathbf{r})\mathbf{H}_t + \hat{\mathbf{D}}_{\mathbf{E}}(\mathbf{H}, \mathbf{E}, \mathbf{r})\mathbf{E}_t$. Hence, for nonlinear, nonhomogeneous and bianisotropic media the pseudo-Riemannian metric (23) is to be replaced by

$$\mathbf{M}(\mathbf{H}, \mathbf{E}, \mathbf{r}) = \begin{pmatrix} -\hat{\mathbf{B}}_{\mathbf{H}}(\mathbf{H}, \mathbf{E}, \mathbf{r}) & -\hat{\mathbf{B}}_{\mathbf{E}}(\mathbf{H}, \mathbf{E}, \mathbf{r}) \\ \hat{\mathbf{D}}_{\mathbf{H}}(\mathbf{H}, \mathbf{E}, \mathbf{r}) & \hat{\mathbf{D}}_{\mathbf{E}}(\mathbf{H}, \mathbf{E}, \mathbf{r}) \end{pmatrix}, \tag{39}$$

which in general loses its interpretation of a (pseudo-Riemannian) metric. The closest analogy with respect to the original metric (2) is the case of nonlinear homogeneous anisotropic media, i.e.,

$$\mathbf{M}(\mathbf{H}, \mathbf{E}) = \begin{pmatrix} -\mu(\mathbf{H}) & \mathbf{0} \\ \mathbf{0} & \varepsilon(\mathbf{E}) \end{pmatrix}, \tag{40}$$

where $\mu(\mathbf{H}) = \hat{\mathbf{B}}_{\mathbf{H}}(\mathbf{H})$ and $\varepsilon(\mathbf{E}) = \hat{\mathbf{D}}_{\mathbf{E}}(\mathbf{E})$ are both symmetric and positive definite.

4. A stability argument

As briefly outlined in the introduction, the principal application of the concept of the mixed-potential is its use in stability theory. To show how the technique of Brayton and Moser can be adapted for electromagnetic systems, let us consider the simplest case in which $\mathbf{B} = \mu \mathbf{H}, \mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{J} = \sigma \mathbf{E}$. Furthermore, the boundary conditions are assumed to be zero. In this case, the mixed-potential functional takes the form

$$\mathcal{P}(\mathbf{H}, \mathbf{E}) = \mathcal{N}(\mathbf{H}, \mathbf{E}) - \int_{\mathbb{V}} \frac{1}{2} \sigma \|\mathbf{E}\|^2 \, \mathrm{d}v, \tag{41}$$

where $\mathcal{N}(\mathbf{H}, \mathbf{E})$ is chosen from (19) and (20). Now, taking the time derivative along the trajectories of (22) yields the indefinite quadratic form

$$\dot{\mathcal{P}}(\mathbf{H}, \mathbf{E}) = -\int_{\mathbb{V}} (\mu \|\mathbf{H}_t\|^2 - \varepsilon \|\mathbf{E}_t\|^2)\, \mathrm{d}v. \tag{42}$$

In the quasi-static case the displacement current $\varepsilon \mathbf{E}_t$ is negligible [10]. This implies $\mathbf{E} = \frac{1}{\sigma}$ curl $\mathbf{H}$, so that the mixed-potential (41) can be rewritten as

$$\mathcal{P}(\mathbf{H}, \mathbf{E}) = \int_{\mathbb{V}} \frac{1}{2\sigma} \|\mathrm{curl}\, \mathbf{H}\|^2\, \mathrm{d}v, \tag{43}$$

and the quadratic form (42) becomes negative definite. This, in turn, implies that (43) is non-increasing which might be used to indicate the system is stable.

In the more general case, however, we need to invoke alternative pairs, $\widetilde{\mathcal{P}}$ and $\widetilde{\mathbf{M}}$, such that the dynamics of (22) are preserved, and such that the symmetric part of $\widetilde{\mathbf{M}}$ is at least negative semi-definite. This is accomplished as follows. Let λ be an arbitrary constant and $\mathbf{K}$ be a (not necessarily constant) symmetric 6×6 matrix, then the construction of an admissible family of pairs starts from

$$\widetilde{\mathcal{P}}(\mathbf{u}) = \lambda \mathcal{P}(\mathbf{u}) + \int_{\mathbb{V}} \frac{1}{2} \delta_{\mathbf{u}} \mathcal{P}(\mathbf{u}) \cdot \mathbf{K} \delta_{\mathbf{u}} \mathcal{P}(\mathbf{u})\, \mathrm{d}v. \tag{44}$$

Indeed, from the latter we have

$$\dot{\widetilde{\mathcal{P}}}(\mathbf{u}) = \int_{\mathbb{V}} \mathbf{u}_t \cdot \lambda \mathbf{M} \mathbf{u}_t\, \mathrm{d}v + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{V}} \frac{1}{2} \delta_{\mathbf{u}} \mathcal{P}(\mathbf{u}) \cdot \mathbf{K} \delta_{\mathbf{u}} \mathcal{P}(\mathbf{u})\, \mathrm{d}v.$$

Now, letting $\mathbf{N} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{0} \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} \mathrm{curl} & \mathbf{0} \\ \mathbf{0} & \mathrm{curl} \end{pmatrix}$, we may write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{V}} \frac{1}{2} \delta_{\mathbf{u}} \mathcal{P}(\mathbf{u}) \cdot \mathbf{K} \delta_{\mathbf{u}} \mathcal{P}(\mathbf{u})\, \mathrm{d}v &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{V}} \frac{1}{2} (\mathbf{NTu} - \bar{\mathcal{G}}_{\mathbf{u}}) \cdot \mathbf{K} (\mathbf{NTu} - \bar{\mathcal{G}}_{\mathbf{u}})\, \mathrm{d}v \\ &= \int_{\mathbb{V}} (\mathbf{NTu}_t - \bar{\mathcal{G}}_{\mathbf{uu}} \mathbf{u}_t) \cdot \mathbf{KMu}_t\, \mathrm{d}v. \end{aligned}$$

Since $\mathbf{NKM}$ is constant by assumption, we obtain

$$\dot{\widetilde{\mathcal{P}}}(\mathbf{u}) = \int_{\mathbb{V}} \mathbf{u}_t \cdot \underbrace{(\lambda \mathbf{M} + \mathbf{NKMT} - \bar{\mathcal{G}}_{\mathbf{uu}} \mathbf{KM})}_{\widetilde{\mathbf{M}}} \mathbf{u}_t\, \mathrm{d}v.$$

The final step is to find among the allowable values for λ and $\mathbf{K}$ a choice which ensures

$$\dot{\widetilde{\mathcal{P}}}(\mathbf{u}) = \int_{\mathbb{V}} \mathbf{u}_t \cdot \widetilde{\mathbf{M}} \mathbf{u}_t\, \mathrm{d}v \leqslant 0. \tag{45}$$

As an example, let us consider the selection $\lambda = 1$ and $\mathbf{K} = \mathrm{diag}(\mathbf{0}, 2\sigma^{-1} \mathbf{I}_3)$. This yields a new mixed-potential functional of the form (cf (43))

$$\widetilde{\mathcal{P}}(\mathbf{H}, \mathbf{E}) = \int_{\mathbb{V}} \frac{1}{2\sigma} \|\mathrm{curl}\, \mathbf{H} - \sigma \mathbf{E}\|^2\, \mathrm{d}v + \int_{\mathbb{V}} \frac{1}{2\sigma} \|\mathrm{curl}\, \mathbf{H}\|^2\, \mathrm{d}v \tag{46}$$

and an (operator) matrix

$$\widetilde{\mathbf{M}} = \begin{pmatrix} -\mu \mathbf{I}_3 & \frac{2\varepsilon}{\sigma}\, \mathrm{curl} \\ \mathbf{0} & -\varepsilon \mathbf{I}_3 \end{pmatrix}. \tag{47}$$

(Note that the latter does not qualify as a metric since it is not symmetric.) However, if the symmetric part of $\widetilde{\mathbf{M}}$ is negative semi-definite, the mixed-potential (46) will be non-increasing along the trajectories. This is accomplished if $\sigma^{-1} \sqrt{\varepsilon \mu^{-1}} \|\mathrm{curl}\| \leqslant 1$.

Another interesting choice is the selection $\lambda = 0$ and $\mathbf{K} = \mathrm{diag}(\mu \mathbf{I}_3, \varepsilon \mathbf{I}_3)^{-1}$. This yields the mixed-potential

$$\widetilde{\mathcal{P}}(\mathbf{H}, \mathbf{E}) = \int_{\mathbb{V}} \frac{1}{2\mu} \|\mathrm{curl}\, \mathbf{E}\|^2 \, \mathrm{d}v + \int_{\mathbb{V}} \frac{1}{2\varepsilon} \|\mathrm{curl}\, \mathbf{H} - \sigma \mathbf{E}\|^2 \, \mathrm{d}v$$
$$= \int_{\mathbb{V}} \frac{1}{2} (\mu \|\mathbf{H}_t\|^2 + \varepsilon \|\mathbf{E}_t\|^2) \, \mathrm{d}v \tag{48}$$

and the (operator) matrix

$$\widetilde{\mathbf{M}} = \begin{pmatrix} \mathbf{0} & \mathrm{curl} \\ -\mathrm{curl} & -\sigma \mathbf{I}_3 \end{pmatrix}, \tag{49}$$

which in turn implies

$$\dot{\widetilde{\mathcal{P}}}(\mathbf{H}, \mathbf{E}) = -\int_{\mathbb{V}} \sigma \|\mathbf{E}_t\|^2 \, \mathrm{d}v \leqslant 0. \tag{50}$$

Observe the resemblance with Poynting's theorem by noting that for the case considered here the total electromagnetic (co-)energy equals

$$\mathcal{E}^*(\mathbf{H}, \mathbf{E}) = \int_{\mathbb{V}} \frac{1}{2} (\mu \|\mathbf{H}\|^2 + \varepsilon \|\mathbf{E}\|^2) \, \mathrm{d}v \tag{51}$$

and (12) reduces to

$$\dot{\mathcal{E}}^*(\mathbf{H}, \mathbf{E}) = -\int_{\mathbb{V}} \sigma \|\mathbf{E}\|^2 \, \mathrm{d}v \leqslant 0. \tag{52}$$

The main difference is that in (12) we start from the electromagnetic energy, whereas the electromagnetic mixed-potential is associated with the power in the system. Since nonzero boundary conditions are easily included by replacing in (44) $\delta_{\mathbf{u}}\mathcal{P}$ by $\delta_{[\mathbf{u}]}\mathcal{P}$, the procedure outlined above can be used to derive a collection of Poynting-like theorems along the lines of [13].

5. Lagrangian boundary control system formulation

In this section, a family of variational principles is proposed which, in contrast to the classical Lagrangian and Hamiltonian formulation, yields both Maxwell's curl equations explicitly. The construction is directly motivated by the form of the electromagnetic mixed-potential functional as defined in section 3. The resulting equation sets can be considered as the infinite-dimensional analogues of the so-called affine Lagrangian and Hamiltonian control systems [6, 21]. The introduction of an appropriate set of conjugate momenta ensures that the corresponding *symplectic* Hamiltonian still coincides with the total stored electromagnetic field energy for which its time derivative precisely coincides with Poynting's theorem. An additional advantage of these new formulations is that there is no need to introduce the usual magnetic vector and electric scalar potentials. This widens the range of applications in which it is, for example, desirable to include fictitious magnetic charges.

5.1. Alternative Lagrangians

For the ease of presentation, we assume that all fields are null at $t \leqslant 0$, and again first consider the situation that $\mathbf{J} = \mathbf{0}$. As shown in section 3, for zero boundary conditions Faraday's law can be expressed as

$$\mu \mathbf{H}_t + \delta_{\mathbf{H}} \int_{\mathbb{V}} \mathbf{H} \cdot \mathrm{curl}\, \mathbf{E} \, \mathrm{d}v = \mathbf{0}.$$

The key idea now is to introduce an integrated magnetic 'displacement' vector

$$\mathbf{Q}(\boldsymbol{r}, t) = \int_{\mathbb{T}} \mathbf{H}(\mathbf{r}, \tau)\, \mathrm{d}\tau, \tag{53}$$

with $\mathbb{T} = [0, t]$ for $t \geqslant 0$, and an integrated electric 'flux' density vector

$$\mathbf{P}(\boldsymbol{r}, t) = \int_{\mathbb{T}} \mathbf{E}(\mathbf{r}, \tau)\, \mathrm{d}\tau. \tag{54}$$

Since $\mathbf{H} = \mathbf{Q}_t$ and $\mathbf{E} = \mathbf{P}_t$, we then may also write

$$\mu \mathbf{Q}_{tt} + \delta_{\mathbf{Q}} \int_{\mathbb{V}} \mathbf{Q} \cdot \operatorname{curl} \mathbf{P}_t \, \mathrm{d}v = \mu \mathbf{Q}_{tt} + \left(\delta_{\mathbf{Q}_t} \int_{\mathbb{V}} \mathbf{Q}_t \cdot \operatorname{curl} \mathbf{P} \, \mathrm{d}v \right)_t = \mathbf{0}, \tag{55}$$

where we exploited the fact that $(\operatorname{curl} \mathbf{P})_t = \operatorname{curl} \mathbf{P}_t$ (Clairaut's theorem). In a similar way, we deduce from

$$\varepsilon \mathbf{E}_t - \delta_{\mathbf{E}} \int_{\mathbb{V}} \mathbf{H} \cdot \operatorname{curl} \mathbf{E} \, \mathrm{d}v = \mathbf{0},$$

the following alternative expressions for Ampère–Maxwell's law:

$$\varepsilon \mathbf{P}_{tt} - \left(\delta_{\mathbf{P}_t} \int_{\mathbb{V}} \mathbf{Q} \cdot \operatorname{curl} \mathbf{P}_t \, \mathrm{d}v \right)_t = \varepsilon \mathbf{P}_{tt} - \delta_{\mathbf{P}} \int_{\mathbb{V}} \mathbf{Q}_t \cdot \operatorname{curl} \mathbf{P} \, \mathrm{d}v = \mathbf{0}. \tag{56}$$

These observations put us in a position to define a Lagrangian functional of the form

$$\mathcal{L}(\bullet, \mathbf{Q}_t, \mathbf{P}_t) = \int_{\mathbb{V}} \left(\frac{1}{2} \mu \|\mathbf{Q}_t\|^2 + \frac{1}{2} \varepsilon \|\mathbf{P}_t\|^2 \right) \mathrm{d}v + \mathcal{L}^{\mathrm{c}}(\bullet), \tag{57}$$

where

$$\mathcal{L}^{\mathrm{c}}(\bullet) = \int_{\mathbb{V}} \bar{\mathcal{L}}^{\mathrm{c}}(\bullet) \, \mathrm{d}v \tag{58}$$

is denoted as the coupling Lagrangian, which is the Lagrangian counterpart of (19). However, instead of the two choices in (20), we now have four possibilities, each accompanied with its particular condition on the boundary, i.e.,

$$\bar{\mathcal{L}}^{\mathrm{c}}(\bullet) = \begin{cases} \mathbf{Q}_t \cdot \operatorname{curl} \mathbf{P} \\ -\operatorname{curl} \mathbf{Q} \cdot \mathbf{P}_t \\ \operatorname{curl} \mathbf{Q}_t \cdot \mathbf{P} \\ -\mathbf{Q} \cdot \operatorname{curl} \mathbf{P}_t . \end{cases} \tag{59}$$

Indeed, the first variation of (57) for all four possibilities of $\bar{\mathcal{L}}^{\mathrm{c}}(\bullet)$ yields the Euler–Lagrange equations

$$\mu \mathbf{Q}_{tt} + \operatorname{curl} \mathbf{P}_t = \mathbf{0}, \qquad \varepsilon \mathbf{P}_{tt} - \operatorname{curl} \mathbf{Q}_t = \mathbf{0}, \tag{60}$$

which upon substitution of $\mathbf{Q}_t = \mathbf{H}$ and $\mathbf{P}_t = \mathbf{E}$ explicitly restores to Maxwell's curl equations (13) and (14), respectively. Again explicit use is made of the fact that all variations vanish at the boundary. This condition will be relaxed in subsection 5.3.

Remark 3. Note that it is also possible to construct suitable combinations of the coupling densities (59). For instance, variation of (57), together with the coupling Lagrangian

$$\mathcal{L}^{\mathrm{c}}(\mathbf{Q}, \mathbf{P}, \mathbf{Q}_t, \mathbf{P}_t) = \frac{1}{2} \int_{\mathbb{V}} \mathbf{Q}_t \cdot \operatorname{curl} \mathbf{P} \, \mathrm{d}v - \frac{1}{2} \int_{\mathbb{V}} \mathbf{P}_t \cdot \operatorname{curl} \mathbf{Q} \, \mathrm{d}v \tag{61}$$

also yields (60). Interestingly, Lagrangians of this type are closely related to the formalism proposed in [24] for lumped electrical RLC circuits.

Before we continue, let us introduce the generalized displacement vector, $\mathbf{q} = \operatorname{col}(\mathbf{Q}, \mathbf{P})$, and its associated generalized velocity vector, $\mathbf{q}_t = \operatorname{col}(\mathbf{Q}_t, \mathbf{P}_t)$. Hence, the Euler–Lagrange equations (60) can be compactly and generally written as

$$\left(\delta_{\mathbf{q}_t} \mathcal{L}(\mathbf{q}, \mathbf{q}_t) \right)_t - \delta_{\mathbf{q}} \mathcal{L}(\mathbf{q}, \mathbf{q}_t) = \mathbf{0}. \tag{62}$$

5.2. Some physical arguments

As argued in [17], there should be a distinction between a so-called mathematical and a physical Lagrangian. In contrast to the former, a physical Lagrangian for a (non-relativistic) system should not only yield the Euler–Lagrange equations leading to the correct equations of motion, but it should also be expressible as the difference between the kinetic (co-)energy and the potential energy. Although the proposed Lagrangian (57) yields the correct equations of motion its form does not equal the usual difference between the magnetic (co-)energy and the electric energy, or vice versa. Instead, it consists of the total stored electromagnetic co-energy plus a coupling term that captures the interaction between the electric and the magnetic field. Interestingly, the coupling Lagrangian (58), with its possible densities (59), corresponds to the class of so-called non-energic Lagrangians [2]. A non-energic Lagrangian is characterized by the fact that it describes neither energy storage nor energy dissipation, which is evident from the observation that the Hamiltonian associated with (58) equals

$$\mathcal{H}^{c}(\bullet) = \int_{\mathbb{V}} \mathbf{q}_t \cdot \delta_{\mathbf{q}_t} \mathcal{L}^{c}(\bullet)\, \mathrm{d}v - \mathcal{L}^{c}(\bullet) \equiv 0. \tag{63}$$

Moreover, the coupling Lagrangian (58) is precisely the electromagnetic analogue of a traditor of the second degree [8]. This, in turn, suggests that the coupling Lagrangian (58) describes the effect of an electromagnetic transformer or gyrator having the curl operator as 'turns ratio' or 'gyration ratio', respectively. We also note that a general traditor of the kth degree admits $2k$ possible Lagrangians. This directly relates to the four possible densities in (59).

In passing on to the full Hamiltonian counterpart, the four Lagrangian coupling densities (59) provide two possible choices for the conjugate momenta. Starting, for example, from

$$\mathcal{L}(\mathbf{P}, \mathbf{Q}_t, \mathbf{P}_t) = \mathcal{E}^*(\mathbf{Q}_t, \mathbf{P}_t) + \begin{cases} \displaystyle\int_{\mathbb{V}} \mathbf{Q}_t \cdot \operatorname{curl} \mathbf{P}\, \mathrm{d}v \\ \displaystyle\int_{\mathbb{V}} \operatorname{curl} \mathbf{Q}_t \cdot \mathbf{P}\, \mathrm{d}v, \end{cases} \tag{64}$$

with

$$\mathcal{E}^*(\mathbf{Q}_t, \mathbf{P}_t) = \int_{\mathbb{V}} \left(\frac{1}{2}\mu \|\mathbf{Q}_t\|^2 + \frac{1}{2}\varepsilon \|\mathbf{P}_t\|^2 \right) \mathrm{d}v \tag{65}$$

provides the following set of conjugate momenta:

$$\mathbf{p} = \delta_{\mathbf{q}_t} \mathcal{L}(\mathbf{q}, \mathbf{q}_t) = \begin{pmatrix} \mathbf{\Lambda} \\ \mathbf{\Gamma} \end{pmatrix} = \begin{pmatrix} \mu \mathbf{Q}_t + \operatorname{curl} \mathbf{P} \\ \varepsilon \mathbf{P}_t \end{pmatrix}. \tag{66}$$

Hence, the Legendre transformation of (64) yields the Hamiltonian

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = \int_{\mathbb{V}} \left(\frac{1}{2\mu} \|\mathbf{\Lambda} - \operatorname{curl} \mathbf{P}\|^2 + \frac{1}{2\varepsilon} \|\mathbf{\Gamma}\|^2 \right) \mathrm{d}v, \tag{67}$$

which clearly coincides with the total electromagnetic (co-)energy (65) since $\mathbf{\Lambda} - \operatorname{curl} \mathbf{P} = \mu \mathbf{Q}_t$ and $\mathbf{\Gamma} = \varepsilon \mathbf{P}_t$. Furthermore, the Hamiltonian is conserved since along the trajectories of

$$\mathbf{q}_t = \delta_{\mathbf{p}} \mathcal{H}(\mathbf{q}, \mathbf{p}) = \begin{pmatrix} \dfrac{1}{\mu}(\mathbf{\Lambda} - \operatorname{curl} \mathbf{P}) \\ \dfrac{1}{\varepsilon} \mathbf{\Gamma} \end{pmatrix} \tag{68}$$

and

$$\mathbf{p}_t = -\delta_{\mathbf{q}} \mathcal{H}(\mathbf{q}, \mathbf{p}) = \begin{pmatrix} \mathbf{0} \\ \dfrac{1}{\mu}(\operatorname{curl} \mathbf{\Lambda} - \operatorname{curl}(\operatorname{curl} \mathbf{P})) \end{pmatrix}, \tag{69}$$

its time derivative satisfies $\dot{\mathcal{H}}(\mathbf{q}, \mathbf{p}) = 0$.

Similarly, starting from

$$\mathcal{L}(\mathbf{Q},\mathbf{Q}_t,\mathbf{P}_t)=\mathcal{E}^*(\mathbf{Q}_t,\mathbf{P}_t)-\begin{cases}\int_{\mathbb{V}}\operatorname{curl}\mathbf{Q}\cdot\mathbf{P}_t\,\mathrm{d}v\\ \int_{\mathbb{V}}\mathbf{Q}\cdot\operatorname{curl}\mathbf{P}_t\,\mathrm{d}v\end{cases} \tag{70}$$

yields

$$\mathbf{p}=\delta_{\mathbf{q}_t}\mathcal{L}(\mathbf{q},\mathbf{q}_t)=\begin{pmatrix}\boldsymbol{\Lambda}\\ \boldsymbol{\Gamma}\end{pmatrix}=\begin{pmatrix}\mu\mathbf{Q}_t\\ \varepsilon\mathbf{P}_t-\operatorname{curl}\mathbf{Q}\end{pmatrix}, \tag{71}$$

and

$$\mathcal{H}(\mathbf{q},\mathbf{p})=\int_{\mathbb{V}}\left(\frac{1}{2\mu}\|\boldsymbol{\Lambda}\|^2+\frac{1}{2\varepsilon}\|\boldsymbol{\Gamma}+\operatorname{curl}\mathbf{Q}\|^2\right)\mathrm{d}v, \tag{72}$$

which also coincides with (65) and satisfies $\dot{\mathcal{H}}(\mathbf{q},\mathbf{p})=0$.

5.3. Free boundary Lagrangian

In the previous subsections, we have explicitly assumed that the variations vanish at the boundary, which, recalling the discussion in section 3, may lead to rather unphysical boundary conditions. In a similar fashion as before, such situations can be avoided by extending the Lagrangian functional with additional boundary terms. For that, let $[\mathbf{q}]=\operatorname{col}(\mathbf{q},\mathbf{q}^{\mathrm{b}})$ and $[\mathbf{q}_t]=\operatorname{col}(\mathbf{q}_t,\mathbf{q}_t^{\mathrm{b}})$, where $\mathbf{q}^{\mathrm{b}}$ and $\mathbf{q}_t^{\mathrm{b}}$ represent the field 'displacements' and 'velocities' at the boundary surface, respectively. The total Lagrangian then takes the form

$$\mathcal{L}[\mathbf{q},\mathbf{q}_t,\mathbf{e}]=\mathcal{L}^{\mathrm{o}}(\mathbf{q},\mathbf{q}_t)+\mathcal{L}^{\mathrm{j}}(\mathbf{q},\mathbf{e})+\mathcal{L}^{\mathrm{b}}(\mathbf{q}^{\mathrm{b}},\mathbf{e}^{\mathrm{b}}), \tag{73}$$

with internal Lagrangian $\mathcal{L}^{\mathrm{o}}(\mathbf{q},\mathbf{q}_t)$ (i.e., the Lagrangian (57)), interaction-source Lagrangian

$$\mathcal{L}^{\mathrm{j}}(\mathbf{q},\mathbf{e})=\int_{\mathbb{V}}\bar{\mathcal{L}}^{\mathrm{j}}(\mathbf{q},\mathbf{e})\,\mathrm{d}v \tag{74}$$

and the interaction-boundary Lagrangian

$$\mathcal{L}^{\mathrm{b}}(\mathbf{q}^{\mathrm{b}},\mathbf{e}^{\mathrm{b}})=\int_{\partial\mathbb{V}}\bar{\mathcal{L}}^{\mathrm{b}}(\mathbf{q}^{\mathrm{b}},\mathbf{e}^{\mathrm{b}})\,\mathrm{d}s. \tag{75}$$

The vector $[\mathbf{e}]=\operatorname{col}(\mathbf{e},\mathbf{e}^{\mathrm{b}})$ represents the independent sources and external fields. This leads to the Euler–Lagrange equations

$$(\delta_{[\mathbf{q}_t]}\mathcal{L}[\mathbf{q},\mathbf{q}_t,\mathbf{e}])_t-\delta_{[\mathbf{q}]}\mathcal{L}[\mathbf{q},\mathbf{q}_t,\mathbf{e}]=\mathbf{0}, \tag{76}$$

which is the infinite-dimensional analogue of an affine Lagrangian boundary control system as originally introduced in [6]. Note that the corresponding total Hamiltonian is followed from taking the Legendre transformation of (73).

As with the definition of the boundary potentials in section 3, the interaction-boundary Lagrangian depends on the choice of the coupling Lagrangian $\mathcal{L}^{\mathrm{c}}(\bullet)$ and can be determined by considering the equations explicitly.

Suppose we start from $\bar{\mathcal{L}}^{\mathrm{c}}=\mathbf{Q}_t\cdot\operatorname{curl}\mathbf{P}$, then the internal Lagrangian takes the form

$$\mathcal{L}^{\mathrm{o}}(\mathbf{P},\mathbf{Q}_t,\mathbf{P}_t)=\mathcal{E}^*(\mathbf{Q}_t,\mathbf{P}_t)+\int_{\mathbb{V}}\mathbf{Q}_t\cdot\operatorname{curl}\mathbf{P}\,\mathrm{d}v, \tag{77}$$

where $\mathcal{E}^*(\mathbf{Q}_t,\mathbf{P}_t)$ represents the electromagnetic (co-)energy defined in (65). If the current density $\mathbf{J}$ is assumed to be defined by independent sources it can be treated as an external quantity, i.e., $\mathbf{e}=-\mathbf{J}$, which enables us to define

$$\mathcal{L}^{\mathrm{j}}(\mathbf{P},\mathbf{J})=-\int_{\mathbb{V}}\mathbf{P}\cdot\mathbf{J}\,\mathrm{d}v. \tag{78}$$

Table 1. Interaction-boundary Lagrangians versus possible coupling Lagrangians.

$\bar{\mathcal{L}}^{\mathrm{c}}$	$\bar{\mathcal{L}}^{\mathrm{b}}$
$\mathbf{Q}_t \cdot \mathrm{curl}\,\mathbf{P}$	$\mathbf{P}\cdot(\mathbf{J}^{\mathrm{s}}+\hat{\mathbf{n}}\times\mathbf{H}^{\mathrm{ext}})$
$-\mathrm{curl}\,\mathbf{Q}\cdot\mathbf{P}_t$	$\mathbf{Q}\cdot\mathbf{E}^{\mathrm{ext}}$
$\mathrm{curl}\,\mathbf{Q}_t\cdot\mathbf{P}$	$\mathbf{Q}\cdot\mathbf{E}^{\mathrm{ext}}$
$-\mathbf{Q}\cdot\mathrm{curl}\,\mathbf{P}_t$	$\mathbf{P}\cdot(\mathbf{J}^{\mathrm{s}}+\hat{\mathbf{n}}\times\mathbf{H}^{\mathrm{ext}})$

Hence, the Lagrangian boundary control system (76) takes the form

$$\begin{pmatrix} \mu\mathbf{Q}_{tt}+\mathrm{curl}\,\mathbf{P}_t \\ \varepsilon\mathbf{P}_{tt} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ \mathrm{curl}\,\mathbf{Q}_t - \mathbf{J} \\ \bar{\mathcal{L}}^{\mathrm{b}}_{\mathbf{Q}} \\ \bar{\mathcal{L}}^{\mathrm{b}}_{\mathbf{P}} - \hat{\mathbf{n}}\times\mathbf{Q}_t \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{79}$$

from which we deduce that $\mathbf{e}^{\mathrm{b}} = \mathbf{H}^{\mathrm{ext}}$ such that

$$\mathcal{L}^{\mathrm{b}}(\mathbf{P},\mathbf{H}^{\mathrm{ext}}) = \int_{\partial\mathbb{V}} \mathbf{P}\cdot(\mathbf{J}^{\mathrm{s}}+\hat{\mathbf{n}}\times\mathbf{H}^{\mathrm{ext}})\,\mathrm{d}s. \tag{80}$$

The interaction-boundary Lagrangians associated with the three remaining choices for $\bar{\mathcal{L}}^{\mathrm{c}}(\bullet)$ are summarized in table 1.

Furthermore, the Hamiltonian associated with (79) reads

$$\mathcal{H}(\mathbf{P},\mathbf{\Lambda},\mathbf{\Gamma},\mathbf{J},\mathbf{H}^{\mathrm{ext}}) = \mathcal{H}^{\mathrm{o}}(\mathbf{P},\mathbf{\Lambda},\mathbf{\Gamma}) + \int_{\mathbb{V}}\mathbf{P}\cdot\mathbf{J}\,\mathrm{d}v - \int_{\partial\mathbb{V}}\mathbf{P}\cdot(\mathbf{J}^{\mathrm{s}}+\hat{\mathbf{n}}\times\mathbf{H}^{\mathrm{ext}})\,\mathrm{d}s, \tag{81}$$

where the internal Hamiltonian $\mathcal{H}^{\mathrm{o}}(\mathbf{P},\mathbf{\Lambda},\mathbf{\Gamma})$ is given by (67). Finally, the time derivative of the internal Hamiltonian yields the energy flow balance

$$\dot{\mathcal{H}}^{\mathrm{o}} = -\int_{\partial\mathbb{V}}\left(\mathbf{P}_t\times\frac{1}{\mu}(\mathbf{\Lambda}-\mathrm{curl}\,\mathbf{P})\right)\cdot\hat{\mathbf{n}}\,\mathrm{d}s - \int_{\mathbb{V}}\mathbf{J}\cdot\mathbf{P}_t\,\mathrm{d}v, \tag{82}$$

which, after substitution of $\mathbf{P}_t = \mathbf{E}$ and $\frac{1}{\mu}(\mathbf{\Lambda}-\mathrm{curl}\,\mathbf{P}) = \mathbf{H}$, constitutes Poynting's theorem (12).

Let us next carry over our findings to the classical Lagrangian and Hamiltonian approach and compare the results.

5.4. Classical approach

Classically the Euler–Lagrange equations associated with an electromagnetic Lagrangian are obtained by invoking a magnetic vector potential $\mathbf{A}$ such that $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$, which requires that $\mathrm{div}\,\mathbf{B} = 0$ (see, e.g. [3, 11, 18]). Substitution of the latter into (6) suggests that the electric field intensity can be written as $\mathbf{E} = -\mathbf{A}_t - \mathrm{grad}\,\phi$, where ϕ represents the electric scalar potential. Indeed, the variation of the Lagrangian functional

$$\mathcal{L}(\phi,\mathbf{A},\mathbf{A}_t) = \int_{\mathbb{V}}\left(\frac{1}{2}\varepsilon\|\mathbf{E}\|^2 - \frac{1}{2\mu}\|\mathbf{B}\|^2\right)\mathrm{d}v + \int_{\mathbb{V}}(\mathbf{J}\cdot\mathbf{A}-\rho\phi)\,\mathrm{d}v, \tag{83}$$

with respect to the magnetic vector potential $\mathbf{A}$ yields, under the condition that

$$\hat{\mathbf{n}}\times\frac{1}{\mu}\mathrm{curl}\,\mathbf{A} = \hat{\mathbf{n}}\times\mathbf{H} = \mathbf{0}, \tag{84}$$

the Ampère–Maxwell law (cf (7)):

$$\varepsilon(\mathbf{A}_t + \operatorname{grad}\phi)_t + \frac{1}{\mu}\operatorname{curl}(\operatorname{curl}\mathbf{A}) + \mathbf{J} = \mathbf{0}.$$

Similarly, variation of the electric potential ϕ yields Gauss' law (cf (8))

$$\varepsilon\operatorname{div}(\mathbf{A}_t + \operatorname{grad}\phi) + \rho = \mathbf{0},$$

together with the boundary condition

$$-\varepsilon(\mathbf{A}_t + \operatorname{grad}\phi)\cdot\hat{\mathbf{n}} = \hat{\mathbf{n}}\cdot\mathbf{D} = 0.$$

The Hamiltonian counterpart is then obtained by defining the momentum conjugate

$$\mathbf{\Pi} = \delta_{\mathbf{A}_t}\mathcal{L}(\phi, \mathbf{A}, \mathbf{A}_t) = \varepsilon(\mathbf{A}_t + \operatorname{grad}\phi) = -\mathbf{D},$$

and considering the Legendre transform $\mathcal{H}(\phi, \mathbf{A}, \mathbf{\Pi})$ of $\mathcal{L}(\phi, \mathbf{A}, \mathbf{A}_t)$ as

$$\begin{aligned}\mathcal{H}(\phi, \mathbf{A}, \mathbf{\Pi}) &= \int_{\mathbb{V}} \mathbf{\Pi}\cdot\mathbf{A}_t\,\mathrm{d}v - \mathcal{L}(\phi, \mathbf{A}, \mathbf{A}_t)\\ &= \int_{\mathbb{V}} \left(\frac{1}{2\varepsilon}\|\mathbf{\Pi}\|^2 + \frac{1}{2\mu}\|\operatorname{curl}\mathbf{A}\|^2\right)\mathrm{d}v\\ &\quad - \int_{\mathbb{V}} (\mathbf{J}\cdot\mathbf{A} - \rho\phi + \mathbf{\Pi}\cdot\operatorname{grad}\phi)\,\mathrm{d}v. \end{aligned} \tag{85}$$

Imposing the same boundary condition as before then gives the Hamiltonian equations of motion:

$$\begin{aligned}\mathbf{A}_t &= \delta_{\mathbf{\Pi}}\mathcal{H}(\phi, \mathbf{A}, \mathbf{\Pi}) = \frac{1}{\varepsilon}\mathbf{\Pi} - \operatorname{grad}\phi\\ -\mathbf{\Pi}_t &= \delta_{\mathbf{A}}\mathcal{H}(\phi, \mathbf{A}, \mathbf{\Pi}) = \frac{1}{\mu}\operatorname{curl}(\operatorname{curl}\mathbf{A}) - \mathbf{J},\end{aligned} \tag{86}$$

together with

$$0 = \delta_{\phi}\mathcal{H}(\phi, \mathbf{A}, \mathbf{\Pi}) = \rho + \operatorname{div}\mathbf{\Pi}. \tag{87}$$

The vanishing of the latter term (implying Gauss'law) is due to the fact that the conjugate momentum associated with the electric scalar potential is not defined since $\delta_{\phi_t}\mathcal{L}(\phi, \mathbf{A}, \mathbf{A}_t) = 0$. Consequently, there is also no Hamiltonian equation for ϕ_t. Furthermore, we observe that the second equation represents the Ampère–Maxwell's law, while Faraday's law follows by taking the curl of the first equation. The remaining Maxwell equation (9) was a necessary condition for the construction of an electromagnetic Lagrangian and Hamiltonian functional in the first place, and therefore is implicitly included in the formulation.

The above formulation can considerably be simplified exploiting the gauge freedom to specify $\phi = 0$ (see [1, 18]). In this case, the Hamiltonian (85) reduces to

$$\mathcal{H}(\mathbf{A}, \mathbf{\Pi}) = \int_{\mathbb{V}} \left(\frac{1}{2\varepsilon}\|\mathbf{\Pi}\|^2 + \frac{1}{2\mu}\|\operatorname{curl}\mathbf{A}\|^2 - \mathbf{J}\cdot\mathbf{A}\right)\mathrm{d}v, \tag{88}$$

providing the equations of motion:

$$\begin{aligned}\mathbf{A}_t &= \delta_{\mathbf{\Pi}}\mathcal{H}(\mathbf{A}, \mathbf{\Pi}) = \frac{1}{\varepsilon}\mathbf{\Pi}\\ -\mathbf{\Pi}_t &= \delta_{\mathbf{A}}\mathcal{H}(\mathbf{A}, \mathbf{\Pi}) = \frac{1}{\mu}\operatorname{curl}(\operatorname{curl}\mathbf{A}) - \mathbf{J}.\end{aligned} \tag{89}$$

Since the boundary conditions of the form (84) are rather unphysical we follow the approach outlined in the previous subsection and consider an interaction-boundary Hamiltonian

$$\mathcal{H}^{\mathrm{b}}(\mathbf{A}, \mathbf{H}^{\mathrm{ext}}) = -\int_{\partial\mathbb{V}} \mathbf{A} \cdot (\mathbf{J}^{\mathrm{s}} + \hat{\mathbf{n}} \times \mathbf{H}^{\mathrm{ext}})\, \mathrm{d}s. \tag{90}$$

Further decomposition of (85) into an internal Hamiltonian,

$$\mathcal{H}^{\mathrm{o}}(\mathbf{A}, \mathbf{\Pi}) = \int_{\mathbb{V}} \left(\frac{1}{2\varepsilon} \|\mathbf{\Pi}\|^2 + \frac{1}{2\mu} \|\mathrm{curl}\, \mathbf{A}\|^2 \right) \mathrm{d}v, \tag{91}$$

and an interaction-source Hamiltonian,

$$\mathcal{H}^{\mathrm{j}}(\mathbf{A}, \mathbf{J}) = -\int_{\mathbb{V}} \mathbf{J} \cdot \mathbf{A}\, \mathrm{d}v, \tag{92}$$

now yields for the time derivative of $\mathcal{H}^{\mathrm{o}}$ along the trajectories of (89):

$$\dot{\mathcal{H}}^{\mathrm{o}}(\mathbf{A}, \mathbf{\Pi}) = \int_{\partial\mathbb{V}} \left(\mathbf{A}_t \times \frac{1}{\mu} \mathrm{curl}\, \mathbf{A} \right) \cdot \hat{\mathbf{n}}\, \mathrm{d}s + \int_{\mathbb{V}} \mathbf{J} \cdot \mathbf{A}_t\, \mathrm{d}v. \tag{93}$$

Clearly, the latter energy flow balance coincides with Poynting's theorem (12) since $\mathbf{A}_t = -\mathbf{E}$. However, for a gauge other than $\phi = 0$ the interpretation of the latter energy flow balance seems far from trivial. This problem does not occur in the Lagrangian framework proposed in the previous sections since the formulation is independent of $\mathbf{A}$ and ϕ.

6. Final remarks and outlook

In this paper, we have developed an electromagnetic analogue of the Brayton–Moser formulation by defining an electromagnetic mixed-potential from which Maxwell's curl equations can be derived. Additionally, the form of the electromagnetic mixed-potential gives rise to alternative variational principles that imply the existence of a family of novel Lagrangian functionals without invoking the usual electric and magnetic potentials. This framework might especially be of interest in cases where it is desirable to consider a set of symmetric Maxwell equations, i.e., in cases where $\mathbf{J}^{\mathrm{m}} \neq \mathbf{0}$ and $\mathrm{div}\, \mathbf{B} \neq 0$.

As argued in [3], Maxwell's equations can be partitioned into kinematical equations, Faraday's law (6) and (9), and dynamical equations, the Ampère–Maxwell law (7) and Gauss' law (8). On the other hand, storage of energy is associated with the dynamics of a system. Hence, from a system-theoretic point of view it seems more natural to consider both Maxwell's curl equations (6) and (7) as the dynamical equations since they produce or affect the storage of energy (as is also underscored by Poynting's theorem). The remaining two laws, (8) and (9), are then to be considered as algebraic constraints on the fields, whereas the constitutive relationships between the field intensities and the flux densities reflect the kinematics.

For the ease of presentation we have mainly considered linear electromagnetic systems. A very strong feature of the Brayton–Moser formulation that deserves further exploration is its use in finding Lyapunov functions for electromagnetic systems with nonlinear constitutive relations. The fact that nonlinear dissipation is included in the framework in an intrinsic manner can be considered as an advantage in contrast to the Hamiltonian-based stability theory. Another, but closely related, feature to investigate further is the construction of alternative Poynting-like flow balances along the lines of [13]. This opens up a way to consider passivity, and possibly stabilization, of electromagnetic systems from a fairly different perspective.

References

[1] Abraham R, Marsden J E and Ratiu T 1988 *Manifolds, Tensor Analysis, and Applications* vol 75 2nd edn (New York: Springer)
[2] Birkhoff G D 1927 *Dynamical Systems* (Providence, RI: American Mathematical Society)
[3] Bracken P 2005 Determination of the electromagnetic Lagrangian from a system of Poisson brackets *Int. J. Theor. Phys.* **44** 127–38
[4] Brayton R K and Miranker W L 1964 A stability theory for nonlinear mixed-initial boundary value problems *Arch. Ration. Mech. Anal.* **17** 358–76
[5] Brayton R K and Moser J K 1964 A theory of nonlinear networks: I. *Q. Appl. Math.* **22** 1–33
[6] Brockett R W 1977 Control theory and analytical mechanics *Geometric Control Theory (Lie Groups: History, Frontiers and Applications* vol VII*)* ed C Martin and R Hermann (Brookline, MA: Math. Sci. Press) pp 1–46
[7] Courant R and Hilbert D 1953 *Methods of Mathematical Physics* vol I (New York: Interscience)
[8] Duinker S 1959 Traditors, a new class of non-energetic non-linear network elements *Philips Res. Rep.* **14** 29–51
[9] Georgieva N K and Tan H W 2003 Potential formalisms in electromagnetic-field analysis *IEEE Trans. Microw. Theory Tech.* **51** 1330–8
[10] Hammond P 1986 *Energy Methods in Electromagnetism* (Oxford: Clarendon)
[11] Jackson J D 1998 *Classical Electrodynamics* 3rd edn (New York: Wiley)
[12] Jeltsema D 2005 Modeling and control of nonlinear networks: a power-based perspective *PhD Thesis* Delft University of Technology, The Netherlands
[13] Jeltsema D, Ortega R and Scherpen J M A 2003 On passivity and power-balance inequalities of nonlinear RLC circuits *IEEE Trans. Circuits Syst.* I **50** 1174–9
[14] Jeltsema D and Scherpen J M A 2005 On Brayton and Moser's missing stability theorem *IEEE Trans. on Circuits and Systems II* **52** (9) 550–2
[15] Millar W 1951 Some general theorems for nonlinear systems possessing resistance *Phil. Mag., Ser. 7* **42** 1150–60
[16] Mohson A 1982 Variational formulation of time-dependent electromagnetic problems *J. Phys. A: Math. Gen.* **15** 1001–6
[17] Morgan M J 1988 Lagrangian formulation of a transmission line *Am. J. Phys.* **56** 639–43
[18] Morishita K and Kumagai N 1977 Unified approach to the derivation of variational expression of electromagnetic fields *IEEE Trans. Microw. Theory Tech.* **25** 34–40
[19] Moser J K 1960 Bistable systems of differential equations with applications to tunnel diode circuits *IBM J. Res. Dev.* **5** 226–40
[20] Nicolet A and Zolla F 2006 On the definition of energy in general electromagnetic media *COMPEL* **25** 55–63
[21] Nijmeijer H and Van der Schaft A J 1990 *Nonlinear Dynamical Control Systems* (New York: Springer)
[22] Olver P J 1993 *Applications of Lie Groups to Differential Equations (Graduate Texts in Mathematics* vol 107*)* 2nd edn (New York: Springer)
[23] Ortega R, Jeltsema D and Scherpen J M A 2003 Power shaping: a new paradigm for stabilization of nonlinear RLC circuits *IEEE Trans. Autom. Control.* **48** 1762–7
[24] Weiss L and Mathis W 1997 A Hamiltonian formulation for complete nonlinear RLC-networks *IEEE Trans. Circuits Syst.* **44** 843–6